NUMERICAL ANALYSIS OF A FREE-BOUNDARY SINGULAR CONTROL PROBLEM IN FINANCIAL ECONOMICS

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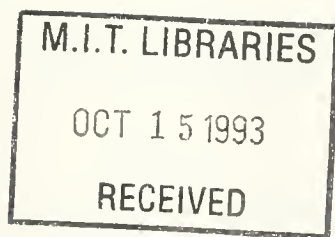
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Abstract

We analyze a numerical scheme for solving the consumption and investment allocation problem studied in a companion paper by Hindy, Huang, and Zhu (1993). For this problem, Bellman's equation is a differential inequality involving second order partial differential equation with gradient constraints. The solution involves finding a free–boundary at which consumption occurs. We prove that the value function is the unique viscosity solution to Bellman's equation. We describe a numerical analysis technique based on Markov chain approximation schemes. We approximate the original continuous time problem by a sequence of discrete parameter Markov chains control problems. We prove that the value functions and the optimal investment policies in the sequence of the approximating control problems converge to the value function and the optimal investment policy, if it exists, of the original problem. We also show that the optimal consumption and abstinence regions in the sequence of approximating problems converge to those in the original problem.

1

1   Introduction

We analyze a numerical scheme for solving the problem of optimal life–time consumption and investment allocation studied in a companion paper by Hindy, Huang, and Zhu (1993), denoted HH&Z henceforth. HH&Z study a model that represents, among other things, habit forming preferences over the service flows from irreversible purchases of a durable good whose stock decays over time. We provide a brief statement of a special case of the control problem in section 2. HH&Z contains the complete formulation, the motivation of the study, the economic interpretations of the variables, and a detailed discussion of the numerical solution. In this paper, we only provide the theoretical framework that supports the analysis in HH&Z.

We use dynamic programming to solve the optimal stochastic control for the continuous time problem formulated in HH&Z. The important feature of this problem is that the admissible controls allow for cumulative consumption processes with possible jumps and singular sample paths. Furthermore, the controls do not appear directly in the objective functional. As a result, the associated Bellman's equation, formula (7) in the sequel, takes the form of a differential inequality involving a nonlinear second order partial differential equation with gradient constraint. The differential inequality reveals the "free–boundary" nature of the solution. In particular, consumption occurs only at some boundary surface in the domain of definition of the value function. This free–boundary is the area in the state space at which *both* the second order differential equation and the gradient constraint are equal to zero. Furthermore, the optimal consumption is the amount required to keep the trajectory of the controlled process from crossing the free–boundary. This typically results in optimal consumption, and associated wealth, with singular sample paths; hence the term "singular control".

In section 3.1, we state the differential inequality that the value function solves. HH&Z contains the details of the derivation which is heuristic because the value function is assumed to be twice continuously differentiable. Under the same assumption, Hindy and Huang (1993) and HH&Z show that if a function solves the differential inequality, together with boundary and regularity conditions, then such a function is indeed the

value function. Furthermore, HH&Z provide a verification theorem to check whether a candidate policy is optimal.

In general, it is difficult to establish smoothness of the value function. To do so, one can either rely on analytical results characterizing the regularity of the solution of Bellman's equation or produce a closed form expression for the value function whose regularity can be verified. Neither option is available in our case. In particular, standard results in the theory of nonlinear partial differential equations; see, for example, Friedman (1964) and Ladyzenskaja, Solonnikov, and Ural'ceva (1968), do not provide evidence for the smoothness of the solution to our differential inequality with its gradient constraints. We will show, however, in section 3 that the differential inequality has a solution in the so-called viscosity sense. Furthermore, that solution is indeed the value function of our dynamic program. We will define the notion of viscosity solutions in section 3.

In section 4, we provide the details of the numerical scheme used for obtaining the solutions reported in HH&Z. The numerical scheme we use is based on the Markov chain approximation method introduced by Kushner (1977) and described in details in Kushner and Dupuis (1992). We approximate the original controlled continuous time processes by a sequence of controlled Markov chains. We parameterize the sequence of controlled Markov chains by the size of the possible jump in each chain which we denote by h. The approximating Markov chains are chosen such that, as $h \downarrow 0$, the mean and mean square change per step, under any control, resemble more and more closely the local drift and the local variance, respectively, of the controlled continuous time process. This property is called "local consistency". We also replace the objective functional for the original problem by a sequence of appropriate analogs for the Markov chains.

Given the chosen approximating Markov chains and objective functionals, the approach is to approximate the original continuous time problem by a sequence of discrete parameter Markov decision problems. For each decision problem in the sequence, the corresponding value function satisfies an iterative Bellman's equation. Such iterative equations are easy to program and solve numerically. Furthermore, the sequence of discrete Bellman's equations can be viewed as finite–difference approximations of the original Bellman equation for the continuous time case.

Next, we turn our attention to the convergence of the numerical scheme. In particular, we focus on convergence of the value functions in the sequence of the approximating decision problems to the value function of the original continuous time problem. We use analytical techniques, in particular the notion of viscosity solutions, to address this question. The argument runs, roughly, as follows. We demonstrate that the value function is the *unique* viscosity solution of Bellman's equation. This is achieved in section 3 in two steps. We first show that the value function is a viscosity solution. We next develop a comparison result between sub- and super-solutions, defined in section 3, and use it to establish the uniqueness result.

We then focus on the sequence of Bellman's equations for the approximating Markov chain problems and consider those as finite–difference approximations to the original Bellman's equation. We establish that this sequence of finite–difference approximations is monotone, consistent, and stable. We will define these notions in section 5.1. Using these properties, we show that the sequence of value functions for the approximating Markov decision problems converges to a viscosity solution of the original Bellman's equation. The uniqueness result established in section 3 implies that this limit is the value function for the original program. We present the convergence analysis in section 5. In that section, we also prove that the sequence of optimal investment policies in the approximating decision problems converges to the optimal investment policy in the original problem if the latter indeed exists. Furthermore, we show that the sequence of regions of consumption and abstinence in the approximating problems converge to the corresponding ones in the original problem.

It is perhaps worthwhile to point out that the analysis in this paper pays special attention to the possibilities of jumps and singular sample paths in the admissible consumption policies, and the special structure of Bellman's equation imposed by the presence of the gradient constraints. In particular, the proofs of theorems 1 to 4 require involved arguments to handle the possible jumps in consumption and the gradient constraints. These arguments extend the existing literature on viscosity solutions in dynamic programming as reported, for example, in Fleming and Soner (1993).

2 Statement of the Control Problem

In this section, we state the control problem which is the focus of analysis in this paper. This problem, with constant investment opportunity set and infinite horizon, is a special case of the general problem studied by HH&Z. For a statement of the general formulation, a complete discussion of the motivations for studying this problem, and for the economic interpretation of the variables, we refer the reader to Hindy and Huang (1993) and HH&Z.

2.1 Financial Market

We consider a frictionless securities market with two long lived and continuously traded securities: a stock and a bond. The stock is risky, pays no dividends, and sells for $S(t)$ at time t. The bond is riskless, does not pay dividend, and sells for $B(t) = e^{rt}$ at time t, where r is the constant riskless interest rate.

The price process[1] for the stock is a geometric Brownian Motion given by:[2]

$$S(t) = S(0) + \int_0^t \mu S(s)\,ds + \int_0^t \sigma S(s)\,dB(s) \quad \forall t \geq 0 \quad a.s., \tag{1}$$

where B is one–dimensional standard Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mu > r$ and $\sigma > 0$ are constants. Note that the prices of both the stock and bond are strictly positive with probability one. The available information in the market is given by $\mathbf{F} = \{\mathcal{F}_t; t \in [0, \infty)\}$ where $\mathcal{F}_t$ is the smallest sub–sigma–field of $\mathcal{F}$ with respect to which $\{S(s); 0 \leq s \leq t\}$ is measurable. All processes to be discussed will be adapted to $\mathbf{F}$.[3]

---

[1] A process Y is a mapping $Y: \Omega \times [0, \infty) \to \Re$ that is measurable with respect to $\mathcal{F} \otimes \mathcal{B}([0, \infty))$, the product sigma–field generated by $\mathcal{F}$ and the Borel sigma-field of $[0, \infty)$. For each $\omega \in \Omega, Y(\omega, .): [0, \infty) \to \Re$ is a *sample path* and for each $t \in [0, \infty), Y(., t): \Omega \to \Re$ is a random variable.

[2] The notation *a.s.* denotes statements which are true with probability one. We will use weak relations. Increasing means nondecreasing and positive means nonnegative.

[3] We assume that $\mathcal{F}_t$ is complete in the sense that it contains all the probability zero sets of $\mathcal{F}$. The process Y is said to be adapted to $\mathbf{F}$ if for each $t \in [0, \infty), Y(t)$ is $\mathcal{F}_t$-measurable.

2.2 Admissible Controls

The control variables are a consumption plan and an investment strategy. A consumption plan C is a process whose sample paths are positive, increasing, and right-continuous, with $C(\omega, t)$ denoting the *cumulative* consumption from time 0 to time t in state ω. An investment strategy is one–dimensional process A, where $A(t)$ denotes the proportion of wealth invested in the stock at time t before consumption and trading. A consumption plan C is said to be *financed* by the investment strategy A if, a.s.,

$$W(t) = W(0) + \int_0^t (rW(s) + A(s)(\mu - r)W(s))\, ds - C(t^-) + \int_0^t \sigma A(s)W(s)\, dB(s), \quad \forall t \geq 0, \tag{2}$$

where $W(t)$ is wealth at time t before consumption. Note that the wealth process has left–continuous sample paths and that $W(t^+) = W(t) - \Delta C(t)$.

A consumption plan C and the investment strategy A that finances it are said to be *admissible* if (2) is well-defined[4] and, for all t, $E[C(t)] \leq Q(t)$, with $Q(t)$ some polynomial of t. Denote by $\mathcal{C}$ and $\mathcal{A}$ the space of admissible consumption plans and trading strategies, respectively.

2.3 Objective Functional

The objective functional is defined as an integral, over time, of a utility function defined over the stock of durable, z, and the standard of living, y, which are given, respectively, by

$$z(t) = z(0^-)e^{-\beta t} + \beta \int_{0-}^t e^{-\beta(t-s)}\, dC(s) \quad a.s., \tag{3}$$

$$y(t) = y(0^-)e^{-\lambda t} + \lambda \int_{0-}^t e^{-\lambda(t-s)}\, dC(s) \quad a.s., \tag{4}$$

where $z(0^-) \geq 0$ and $y(0^-) \geq 0$ are constants, β and λ, with $\beta > \lambda$, are weighting factors, and the integrals in (3) and (4) are defined path by path in the Lebesgue-Stieltjes sense.

---

[4]For this we mean both the Lebesgue integral and the Itô integral are well–defined. When $A(t)$ is a feedback control depending on the state variables and $C(t)$ depends on the history of (W, S), we mean there exists a solution W to the stochastic differential equation (2) for every pair of feedback controls.

Wealth is managed dynamically to solve the following program:

$$
\begin{aligned}
\sup_{C \in \mathcal{C}} \quad & E\left[\int_0^\infty e^{-\delta t} u(y(t), z(t)) dt\right] \\
\text{s.t.} \quad & C \text{ is financed by } A \in \mathcal{A} \text{ with } W(0) = W_0, \\
& \text{and } W(t) - \Delta C(t) \geq 0 \quad \forall t \geq 0,
\end{aligned}
\tag{5}
$$

where W_0 is initial wealth, $\delta > 0$ is a discount factor and $u(\cdot, \cdot)$ is the felicity function. The function u is assumed to satisfy the following conditions:

- u is nonnegative and satisfies the linear growth condition:

$$
0 \leq u(y, z) \leq K(y + z), \qquad \forall y \geq 0, \ z \geq 0. \quad \text{for some constant } K > 0. \tag{6}
$$

- For a fixed y, $u(y, \cdot)$ is a strictly increasing, concave, and differentiable function.

- For a fixed z, $u(\cdot, z)$ is a strictly increasing, concave, and differentiable function.

The second constraint in (5) is the requirement that wealth after consumption at any time must be positive. Condition (6) guarantees that the supremum in (5) is finite.

Since the interest rate and the parameters of the return on the stock are constant, the supremum in (5) is a function of the state variables W, y, and z only. As usual, we call the supremum in (5) the *value function* of the dynamic program and denote it by $J(W, y, z)$. We observe that, in general, little is known about the differentiability of J. The choice of a technique to analyze the dynamic program in (5) is influenced by prior knowledge of the smoothness of the value function.

3 Analysis of The Value Function

3.1 Bellman's Equation

In this section, we report Bellman's equation associated with the dynamic program in (5). Although little is known about the regularity of the value function J, we can derive Bellman's equation heuristically by assuming that J is twice continuously differentiable.

This assumption allows us to utilize Itô's lemma, together with the dynamic programming principle, to derive Bellman's equation. Bellman's equation turns out to be a differential inequality involving a nonlinear second order partial differential equation with gradient constraints. In section 3, we prove rigorously that the value function is indeed a solution of Bellman's equation we derive heuristically here. The value function, however, is a solution in the viscosity sense— a notion that we define in section 3.

Assuming that the value function is smooth, HH&Z show that the necessary conditions for optimality are given by the following *differential inequality*:

$$\max\left\{ \max_A[u(y,z) + J_W W[r + A(\mu - r)] + \tfrac{1}{2}J_{WW}W^2A^2\sigma - \beta z J_z - \lambda y J_y - \delta J], \right.$$
$$\left. \beta J_z + \lambda J_y - J_W\right\} = 0, \tag{7}$$

plus the condition that nontrivial consumption occurs only at states where $\beta J_z + \lambda J_y = J_W$. The differential inequality in (7) can also be viewed, in the region of no consumption, as a nonlinear second–order partial differential equation with the gradient constraint that $\beta J_z + \lambda J_y - J_W \leq 0$.

In addition to the above necessary conditions, it is clear that J must satisfy the following *boundary condition*:

$$\Psi(y,z) \equiv \lim_{W \downarrow 0} J(W,y,z) = \int_0^\infty e^{-\delta s}u(ye^{-\lambda s}, ze^{-\beta s})ds. \tag{8}$$

The differential inequality (7) suggests that the dynamic program formulated in section 2 is a free–boundary singular control problem. Consumption should occur only when the state variables (W, y, z) reach a certain boundary surface characterized by the condition that $J_W = \beta J_z + \lambda J_y$. Furthermore, the optimal consumption is the amount required to keep the trajectory of the controlled processes from crossing this critical boundary. Since the returns on the stock follow a Brownian Motion, such a consumption policy, and the associated wealth process, may have singular sample paths— hence the term "singular control". Computing the location of the free–boundary is the focus of this paper.

Hindy and Huang (1993) and HH&Z show that if there is a smooth solution to (7) that satisfies (8) and additional mild regularity conditions, then that solution is the

value function for the program (5). These papers also provide verification theorems that specify conditions for a candidate feasible consumption and investment policy to attain the supremum in (5). We refer the reader to HH&Z (§3) for the statements of the necessary and sufficient conditions of optimality, the verification theorems, and the proofs.

3.2 Viscosity Solutions of The Dynamic Programming Equation

The verification theorem presented in HH&Z is useful when it is known that the value function is twice continuously differentiable. Such is the case when there is a closed form solution of Bellman's equation as in Hindy and Huang (1993). In general, it is difficult to know whether the value function has enough regularity for the verification theorem to apply. To handle the situations in which there is no information about the regularity of the value function, we utilize the notion of weak, or viscosity, solutions.

Viscosity solutions were introduced by Crandall and Lions (1982) for first–order equations, and by Lions (1983) for second–order equations. For a general overview of the theory, we refer the reader to the *User's Guide* by Crandall, Ishii, and Lions (1991).

We are interested in viscosity solutions in the context of proving convergence of the Markov chain numerical scheme to be presented in the following sections. To that end, we characterize the value function as the *unique* viscosity solution of Bellman's equation. We prove uniqueness of the solution to Bellman's equation by utilizing a comparison principle between viscosity subsolutions and viscosity supersolutions. This comparison result has additional significance to our subsequent analysis because it is a key element in establishing convergence of the numerical scheme.

Let $Q \equiv (0,\infty) \times [0,\infty) \times [0,\infty)$ and consider a second–order partial differential equation of the form:

$$\max\left\{-\delta v + F(\partial_W^2 v, Dv) + u(y,z), \vec{\gamma} \cdot D^\mathsf{T} v\right\} = 0, \quad \text{on } Q \quad \text{with } v(0,y,z) = \Phi(y,z), \quad (9)$$

where $\vec{\gamma} = (-1, \lambda, \beta)$, $D = (\partial_W, \partial_y, \partial_z)$, Φ is some function of y and z, and $^\mathsf{T}$ denotes

transpose. The function F is given by:

$$F(\partial_W^2 v, \partial_W v, \partial_y v, \partial_z v) = \max_A \{ \frac{1}{2} W^2 A^2 \sigma^2 \partial_W^2 v + W[r + A(\mu - r)] \partial_W v \} - \lambda y \partial_y v - \beta z \partial_z v.$$

Next, we state the notion of viscosity solutions as in Crandall, Ishii, and Lions (1991).

Definition 1 (Viscosity Solution) *(i) An upper semicontinuous function $v : Q \to R$ is a (viscosity) subsolution of (9) on the domain Q if $v(0, y, z) \leq \Phi(y, z)$ and for every $\phi \in C^{2,1,1}(Q)$ and any local maximum point $(W_0, y_0, z_0) \in Q$ of $v - \phi$,*

$$\max \Big\{ -\delta v(W_0, y_0, z_0) + F(\partial_W^2 \phi(W_0, y_0, z_0), \phi_W(W_0, y_0, z_0), \phi_y(W_0, y_0, z_0), \phi_z(W_0, y_0, z_0))$$
$$+ u(y_0, z_0), \vec{\gamma} \cdot D^\mathsf{T} \phi(W_0, y_0, z_0) \Big\} \geq 0 \,. \tag{10}$$

(ii) A lower semicontinuous function $v : Q \to R$ is a (viscosity) supersolution of (9) on the domain Q if $v(0, y, z) \geq \Phi(y, z)$ and for every $\phi \in C^{2,1,1}(Q)$ and any local minimum point $(W_0, y_0, z_0) \in Q$ of $v - \phi$,

$$\max \Big\{ -\delta v(W_0, y_0, z_0) + F(\partial_W^2 \phi(W_0, y_0, z_0), \phi_W(W_0, y_0, z_0), \phi_y(W_0, y_0, z_0), \phi_z(W_0, y_0, z_0))$$
$$+ u(y_0, z_0), \vec{\gamma} \cdot D^\mathsf{T} \phi(W_0, y_0, z_0) \Big\} \leq 0 \,. \tag{11}$$

A continuous function $v : Q \to R$ is a viscosity solution of (9) if it is both a subsolution and a supersolution of (9).

The main result of this section is that $J(W, y, z)$ is the unique viscosity solution of Bellman's equation (9) with boundary condition $v(0, y, z) = \Psi(y, z)$ as in (8). First, we state without proof the following proposition

Proposition 1 (Optimality Principle) *The value function J satisfies the dynamic programming principle*

$$J(W, y, z) = \sup_{(A,C) \in \mathcal{AC}} E \Big[e^{-\delta \tau} J(W(\tau), y(\tau), z(\tau)) + \int_0^\tau e^{-\delta t} u(y(t), z(t)) dt \Big], \tag{12}$$

for any F_t-adapted stopping time $\tau \geq 0$, where $W(\tau)$, $y(\tau)$, and $z(\tau)$ are, respectively, the stopped processes of W, y, and z at τ.

The dynamic programming principle in proposition 1 is similar to what can be found, for example, in Fleming and Soner (1993, §5.2). The proof of proposition 1 is, however, slightly different from the standard proofs because in our formulation the admissible controls allow for jumps and singular sample paths. A proof of the dynamic programming principle in such a case can be found in Zhu (1991, Chapter II). Next, we show how one may utilize the principle of dynamic programming to prove that the value function is a viscosity solution of Bellman's equation. The proof of this result is not standard because of the possible jumps and the singular sample paths in the admissible controls.

Theorem 1 *The value function J is a viscosity solution of Bellman's equation (9).*

Proof. We first establish that the value function J is continuous on the domain Q. Utilizing the concavity of u, and using standard reasoning, we can show that J is concave in all of its arguments. It then follows that J is continuous in the region $(0, \infty) \times (0, \infty) \times (0, \infty)$; see Holmes (1975, §14, page 82). We then need to show that J is continuous on the boundary $(W, 0, 0)$, $W \in (0, \infty)$, of the doamin Q. Since u is increasing, the value function J is increasing in all of its elements, and hence $\lim_{y \to 0, z \to 0} J(W, y, z) \geq J(W, 0, 0)$. We prove that, in fact, $\lim_{y \to 0, z \to 0} J(W, y, z) = J(W, 0, 0)$.

Suppose that $\lim_{y \to 0, z \to 0} J(W, y, z) > J(W, 0, 0)$. From the dynamic programming principle, we have

$$J(W, 0, 0) \geq E\left[e^{-\delta t} J(\bar{W}_t, \bar{y}_t, \bar{z}_t) + \int_0^t e^{-\delta s} u(\bar{y}_s, \bar{z}_s)\, ds\right], \tag{13}$$

where $(\bar{W}_s, \bar{y}_s, \bar{z}_s)$ are the trajectories of a continuous state process for some admissible controls. Taking the limit as $t \downarrow 0$, and utilizing the continuity of the trajectory of the controlled process, the right-hand side of (13) converges to a value strictly greater than $J(W, 0, 0)$– a clear contradiction.

We next show that J is a viscosity solution of Bellman's equation.
(i) J is a viscosity subsolution. We argue by contradiction and show that if J fails to be a subsolution, then the dynamic programming principle stated in proposition 1 is violated. Let ϕ be a $C^{2,1,1}(Q)$ function such that $J - \phi$ has a local maximum at $(W_0, y_0, z_0) \in Q$.

That is, $\exists$ a neighborhood $\mathcal{N}_0$ of (W_0, y_0, z_0) such that

$$J(W_0, y_0, z_0) - \phi(W_0, y_0, z_0) \geq J(W, y, z) - \phi(W, y, z), \qquad \forall (W, y, z) \in \mathcal{N}_0. \qquad (14)$$

Without loss of generality, we assume that $J(W_0, y_0, z_0) = \phi(W_0, y_0, z_0)$. Now suppose that ϕ violates inequality (10). In other words, both $\lambda \phi_y + \beta \phi_z - \phi_W < 0$ and $-\delta J + F(\phi_{WW}, \phi_W, \phi_y, \phi_z) + u(y, z) < 0$ at (W_0, y_0, z_0). By continuity, and using the assumption that $J(W_0, y_0, z_0) = \phi(W_0, y_0, z_0)$, there exists $\gamma > 0$ such that, for (W, y, z) in some neighborhood $\hat{\mathcal{N}}$ of (W_0, y_0, z_0),

$$\lambda \phi_y(W, y, z) + \beta \phi_z(W, y, z) - \phi_W(W, y, z) < 0 \quad \text{and} \tag{15}$$

$$- \delta \phi(W, y, z) + F(\partial_W^2 \phi(W, y, z), \phi_W(W, y, z), \phi_y(W, y, z), \phi_z(W, y, z)) + u(y, z) < -\gamma. \tag{16}$$

By the existence result in Zhu (1991, Theorems IV.2.2 and IV.2.3), we consider the optimal policy (A^*, C^*) at (W_0, y_0, z_0).[5] We now establish that the optimal policy C^* does not prescribe an initial jump at (W_0, y_0, z_0). If not, then C^* prescribes some jump η_0. Using the dynamic programming principle, we can conclude that

$$J(W_0, y_0, z_0) \leq J(W_0 - \eta, y_0 + \lambda \eta, z_0 + \beta \eta), \qquad \forall 0 \leq \eta < \eta_0.$$

By the property of the local maximum in (14), the previous inequality also holds for ϕ, and hence

$$0 \leq \phi(W_0 - \eta, y_0 + \lambda \eta, z_0 + \beta \eta) - \phi(W_0, y_0, z_0)$$

where $0 < \eta \leq \eta_0$ is sufficiently small so that $(W_0 - \eta, y_0 + \lambda \eta, z_0 + \beta \eta) \in \mathcal{N}_0$. Dividing by η and letting $\eta \downarrow 0$, we obtain

$$0 \leq \lambda \phi_y(W_0, y_0, z_0) + \beta \phi_z(W_0, y_0, z_0) - \phi_W(W_0, y_0, z_0),$$

---

[5] From the analysis in Zhu (1991, Chapter IV), the optimal policy (A^*, C^*) is known to exist. However, (A^*, C^*) may be realized in a space of processes larger than the space of admissible controls $(\mathcal{A}, \mathcal{C})$. The crucial feature of the analysis is that the expected life time utility associated with (A^*, C^*), starting from any point (W, y, z), is equal to the value function J evaluated at that point. For more details, we refer the reader to Zhu (1991).

which is a contradiction to (15).

We have now established that if (15) holds, then the optimally controlled process (W_t^*, y_t^*, z_t^*) is continuous in a neighborhood of (W_0, y_0, z_0). We next show that if J fails to be a subsolution, then the dynamic programming principle is violated.

We define the optional time $\tilde{\tau} = \inf\{t \geq 0 : (W_t^*, y_t^*, z_t^*) \notin \mathcal{N}_0\}$. Clearly, $\tilde{\tau} > 0$, a.s., by the established fact that (W_t^*, y_t^*, z_t^*) is a.s. continuous up to τ under the optimal policy. Then, it follows from (15) and (16) that

$$E\left[\int_0^{\tilde{\tau}} e^{-\delta t}[-\delta\phi + F(\partial_W^2\phi, \phi_W, \phi_y, \phi_z) + u](W_t^*, y_t^*, z_t^*)dt\right] \tag{17}$$

$$+E\left[\int_0^{\tilde{\tau}} e^{-\delta t}[\lambda\phi_y + \beta\phi_z - \phi_W](W_t^*, y_t^*, z_t^*)dC_t^n\right] < -\gamma E[\tilde{\tau}].$$

Define the optional time $\hat{\tau} = \inf\{t \geq 0 : \int_0^t |A_s^* W_s^* \sigma|^2 ds \geq K\}$, for some constant K. Now take τ to be the minimum of $\tilde{\tau}$ and $\hat{\tau}$. Applying Ito's rule to the process $e^{-\delta t}\phi(W_t^*, y_t^*, z_t^*)$, we have

$$\begin{aligned}
\phi(W_0, y_0, z_0) &= E[e^{-\delta\tau}\phi(W_\tau^*, y_\tau^*, z_\tau^*)] \\
&\quad -E\left[\int_0^\tau e^{-\delta t}[-\delta\phi + F(\partial_W^2\phi, \phi_W, \phi_y, \phi_z)](W_t^*, y_t^*, z_t^*)dt\right] \\
&\quad -E\left[\int_0^\tau e^{-\delta t}[\lambda\phi_y + \beta\phi_z - \phi_W](W_t^*, y_t^*, z_t^*)dC_t^*\right].
\end{aligned} \tag{18}$$

Note that the expectation of the stochastic integral $\int_0^\tau A_t^* W_t^* \sigma \, dB_t$ vanishes because that integral is a martingale as the integrand is bounded till the stopping time τ. Again by the local maximum property in (14), we can replace ϕ by J in first line of (18) and then combine the result with (17) to conclude that

$$\begin{aligned}
J(W_0, y_0, z_0) &\geq E\left[e^{-\delta\tau}J(W_\tau^*, y_\tau^*, z_\tau^*)\right] \\
&\quad -E\left[\int_0^\tau e^{-\delta t}[-\delta\phi + F(\partial_W^2\phi, \phi_W, \phi_y, \phi_z)](W_t^*, y_t^*, z_t^*)dt\right] \\
&\quad -E\left[\int_0^\tau e^{-\delta t}[\lambda\phi_y + \beta\phi_z - \phi_W](W_t^*, y_t^*, z_t^*)dC_t^*\right] \\
&\geq E\left[\int_0^\tau e^{-\delta t}u(y_t^*, z_t^*)dt + e^{-\delta\tau}J(W_\tau^*, y_\tau^*, z_\tau^*)\right] + \gamma E[\tau],
\end{aligned}$$

which clearly violates the optimality of (A^*, C^*) and the principle of dynamic programming (12).

(ii) J is a viscosity supersolution. By definition, we must show, for every $C^{2,1,1}$– smooth ϕ such that $J(W_0, y_0, z_0) - \phi(W_0, y_0, z_0) \leq J(W, y, z) - \phi(W, y, z)$ for $(W, y, z) \in \mathcal{N}_0$, that the following holds

$$\left[-\delta J + F(\phi_{WW}, \phi_W, \phi_y, \phi_z) + u\right](W_0, y_0, z_0) \leq 0 \text{ and} \tag{19}$$

$$\lambda \phi_y(W_0, y_0, z_0) + \beta \phi_z(W_0, y_0, z_0) - \phi_W(W_0, y_0, z_0) \leq 0. \tag{20}$$

First, we choose zero consumption $C_s \equiv 0$ and some investment policy A on the stochastic interval $[0, \tau)$, where $\tau = \inf\{t > 0 : \int_0^t |A_s W_s \sigma|^2 ds \geq K\}$ for some constant $K > 0$. Let W_s^0, y_s^0, and z_s^0 be the state processes corresponding to the controls $C_s \equiv 0$ and A on $[0, \tau)$. By the principle of dynamic programming, it follows that, for all $\rho \leq \tau$, where ρ is an optional time,

$$J(W_0, y_0, z_0) \geq E\left[e^{-\delta\rho} J(W_\rho^0, y_\rho^0, z_\rho^0) + \int_0^\rho e^{-\delta s} u(y_s^0, z_s^0) ds\right].$$

Replacing J by ϕ, we then apply Ito's rule, together with the observation that the stochastic integral is a martingale, to obtain from the above inequality that

$$E\left[\int_0^\rho e^{-\delta s}[-\delta\phi + F + u](W_s^0, y_s^0, z_s^0) ds\right] \leq 0 \quad \forall 0 \leq \rho \leq \tau.$$

Dividing by ρ and letting $\rho \downarrow 0$, using the continuity of the integrand, we derive the first inequality (19). To derive (20), the dynamic programming principle implies that

$$J(W_0, y_0, z_0) \geq J(W_0 - \eta, y_0 + \lambda\eta, z_0 + \beta\eta), \qquad \forall 0 \leq \eta \leq W_0.$$

Since $J - \phi$ has a local minimum at (W_0, y_0, z_0), this inequality also holds for ϕ as well if $\eta > 0$ is sufficiently small such that $(W_0 - \eta, y_0 + \lambda\eta, z_0 + \beta\eta) \in \mathcal{N}_0$. Therefore, subtracting the right-hand side from the left-hand side and dividing by η and sending $\eta \downarrow 0$, we obtain the second inequality (20). This completes the proof. ∎

3.3   Comparison of Subsolutions and Supersolutions

We conclude this section by providing a theorem that compares the subsolutions of Bellman's equation to its supersolutions. The proof of this result is not a simple extension of standard results as, for example, in Fleming and Soner (1993, §5.8). We utilize additional arguments to handle the gradient constraint that $J_W - \lambda J_y - \beta J_z \geq 0$ in Bellman's equation. The comparison result is then used to argue that the value function is the *unique* viscosity solution of the dynamic programming equation. In addition, the comparison result is later used in the proof of convergence of the Markov chain approximations to be presented in a subsequent section.

Theorem 2 (Comparison Principle) *Let V and $\bar{V}$ be, respectively, a subsolution and a supersolution of (9) on the domain Q in the sense of Definition 1, then $V \leq \bar{V}$ on the domain Q.*

Proof. Let us denote, for simplicity, $x \equiv (W, y, z)$ and $\bar{x} \equiv (\bar{W}, \bar{y}, \bar{z})$. We rewrite Bellman's equation as $\max\{-\delta v + F(\partial_W^2 v, D_x v) + u(y, z), \vec{\gamma} \cdot D_x^\mathsf{T} v\} = 0$, where $\vec{\gamma} = (-1, \lambda, \beta)$ and D_x^T is the transpose of $(\partial_W, \partial_y, \partial_z)$. We construct a function $U(W, y, z) \equiv \int_0^\infty e^{-\delta t} u(y e^{-\lambda t}, z^{-\beta t}) \, dt - m_0$ for some constant $m_0 > 0$. One can easily verify that

$$-\delta U + F(\partial_W^2 U, D_x U) + u(y, z) > 0 \quad \text{and} \quad \lambda U_y + \beta U_z - U_W > 0,$$

where the second inequality follows from the assumption that u is strictly increasing in both y and z.

Next, we will prove the comparison between $\bar{V}$ and $V^\theta = \theta V + (1 - \theta)U$, for all θ, from which the proof of the theorem is immediate. For this purpose, we choose the auxiliary function

$$\psi(x, \bar{x}) = \frac{\alpha}{2}|x - \bar{x}|^2 + \epsilon(|x|^2 + |\bar{x}|^2), \qquad \forall (x, \bar{x}) \in Q \times Q,$$

where $\alpha > 0$ and $0 < \epsilon < 1$. Assume to the contrary that:

$$V^\theta(x^*) > \bar{V}(x^*), \qquad \text{for some } x^* \in Q. \tag{21}$$

By the linear growth of V and $\bar{V}$, it is easy to check that $V^\theta(x) - \bar{V}(\bar{x}) - \psi(x,\bar{x})$ will attain its maximum at a point $(x_0, \bar{x}_0) \in Q \times Q$. Note that $(x_0, \bar{x}_0)$ depends on the values of α and ϵ. We will suppress this dependence in the notation. One can prove, as in Zhu (1991, Appendix A: proof of comparison principle), that $\alpha |x_0 - \bar{x}_0|^2 \to 0$ as $\alpha \uparrow \infty$, and $\epsilon(|x_0| + |\bar{x}_0|) \to 0$ as $\epsilon \downarrow 0$.

We will show that supposition (21) produces the following contradiction. From the definition of $(x_0, \bar{x}_0)$, we have, for small ϵ,

$$
\begin{aligned}
V^\theta(x_0) - \bar{V}(\bar{x}_0) - \psi(x_0, \bar{x}_0) &= \max_{(x,\bar{x}) \in Q \times Q} V^\theta(x) - \bar{V}(\bar{x}) - \psi(x,\bar{x}) \\
&\geq V^\theta(x^*) - \bar{V}(x^*) - 2\epsilon|x^*|^2 > 0.
\end{aligned}
$$

However, we will show next that the fact that V^θ and $\bar{V}$ are sub- and supersolutions, respectively, implies that $V^\theta(x_0) - \bar{V}(\bar{x}_0) - \psi(x_0, \bar{x}_0) \downarrow 0$ as $\epsilon \downarrow 0$ and $\alpha \uparrow \infty$, which is a clear contradiction.

Applying Definition 1 to the subsolution V^θ and the super-solution $\bar{V}$, we conclude that

$$
\max\{-\delta V^\theta + F(\partial_W^2 \psi, D_x \psi) + u(\cdot), \vec{\gamma} \cdot D_x^{\mathsf{T}} \psi\} > 0, \qquad \text{at } x_0 \qquad (22)
$$

$$
\max\{-\delta \bar{V} + F(-\partial_{\bar{W}}^2 \psi, -D_{\bar{x}} \psi) + u(\cdot), -\vec{\gamma} \cdot D_{\bar{x}}^{\mathsf{T}} \psi\} \leq 0, \qquad \text{at } \bar{x}_0. \qquad (23)
$$

Now, since $D_x \psi = 2\epsilon x + \alpha |x - \bar{x}|$ and $D_{\bar{x}} \psi = 2\epsilon \bar{x} - \alpha |x - \bar{x}|$, it follows from inequality (23) that

$$
-\vec{\gamma} \cdot D_{\bar{x}}^{\mathsf{T}} \psi = -\vec{\gamma} \cdot [2\epsilon \bar{x}_0 - \alpha(x_0 - \bar{x}_0)] \leq 0. \qquad (24)
$$

Moreover, we claim that in (22), $\vec{\gamma} \cdot [2\epsilon x_0 + \alpha(x - \bar{x}_0)] \leq 0$. If not, the opposite inequality holds. Subtracting it from (24), we conclude that $2\epsilon \vec{\gamma} \cdot (\bar{x}_0 + x_0) > 0$, $\forall \alpha > 0$, which is a contradiction to the fact that $\epsilon(|x_0| + |\bar{x}_0|) \downarrow 0$ as $\epsilon \downarrow 0$. Therefore, we must have in (22) and (23) that

$$
-\delta V^\theta(x_0) + F(\partial_W^2 \psi, D_x \psi) + u(y_0, z_0) > 0, \qquad \text{at } x_0 \quad \text{and}
$$

$$
-\delta \bar{V}(\bar{x}) + F(-\partial_{\bar{W}}^2 \psi, -D_{\bar{x}} \psi) + u(\bar{y}_0, \bar{z}_0) \leq 0, \qquad \text{at } \bar{x}_0
$$

Subtracting one from the other yields the inequality

$$\delta[V^\theta(x_0) - \bar{V}(\bar{x}_0)] \le F(\partial^2_W \psi, D_x \psi) - F(-\partial^2_W \psi, -D_{\bar{x}} \psi) + u(y_0, z_0) - u(\bar{y}_0, \bar{z}_0). \quad (25)$$

Meanwhile, we calculate (Note: $r = 0$ for simplicity)

$$
\begin{aligned}
F(\partial^2_W \psi, D_x \psi) &= \max_A \{ \frac{\alpha}{2} W_0^2 A^2 \sigma^2 + W_0 A \mu [\alpha(W_0 - \bar{W}_0) + 2\epsilon W_0] \} \\
&\quad - \alpha[\lambda y_0(y_0 - \bar{y}_0) + \beta z_0(z_0 - \bar{z}_0)] \\
F(-\partial^2_W \psi, -D_{\bar{x}} \psi) &= \max_A \{ -\frac{\alpha}{2} \bar{W}_0^2 A^2 \sigma^2 - \bar{W}_0 A \mu [-\alpha(W_0 - \bar{W}_0) + 2\epsilon \bar{W}_0] \} \\
&\quad - \alpha[\lambda \bar{y}_0(y_0 - \bar{y}_0) + \beta \bar{z}_0(z_0 - \bar{z}_0)]
\end{aligned}
$$

where the maximizers can be explicitly computed as

$$A_0^* = -\frac{\mu}{\sigma^2} \frac{\alpha(W_0 - \bar{W}_0) + 2\epsilon W_0}{\alpha W_0} \quad \text{and} \quad \bar{A}_0^* = -\frac{\mu}{\sigma^2} \frac{-\alpha(W_0 - \bar{W}_0) + 2\epsilon \bar{W}_0}{\alpha \bar{W}_0}.$$

Substituting into (25) and rearranging the terms, we can verify that

$$
\begin{aligned}
\delta[V^\theta(x_0) - \bar{V}(\bar{x}_0)] &\le \frac{-\mu^2}{\sigma^2} \alpha(W_0 - \bar{W}_0)^2 - 2\epsilon \frac{\mu^2}{\sigma^2}(W_0 - \bar{W}_0)^2 - 2\epsilon^2 \frac{\mu^2}{\alpha \sigma^2}(W_0^2 + \bar{W}_0^2) \\
&\quad - \alpha(\lambda|y_0 - \bar{y}_0|^2 + \beta|z_0 - \bar{z}_0|^2) + [u(y_0, z_0) - u(\bar{y}_0, \bar{z}_0)],
\end{aligned}
$$

which goes to zero as $\epsilon \downarrow 0$ and $\alpha \uparrow \infty$. This completes the demonstration of a contradiction and the proof of the theorem. ∎

Corollary 1 *The value function J is the* unique *viscosity solution of Bellman's equation (9).*

Proof. Suppose that there is another viscosity solution V. We then have $V \le J$ because V and J are sub- and supersolutions, respectively. On the other hand, we have $J \le V$ because J and V are also sub- and supersolutions, respectively. Hence, $J = V$. ∎

4   Numerical Approach

In this section we describe the numerical approach utilized to solve the infinite horizon utility maximization problem. We replace the continuous time processes (W, y, z) by a sequence of approximating discrete parameter Markov chains. The important feature of the approximating sequence is that, as the step size converges to zero, the mean and mean square change per step, under any control, converge to the local drift and the local variance, respectively, of the controlled continuous time process. This property is called "local consistency". The original optimization problem is then approximated by a sequence of discrete parameter control problems. We show that the value function of each Markov chain control problem satisfies an iterative, and hence easily programmable, discrete Bellman's equation. The Bellman's equation of the discrete control problem is a natural finite–difference approximation to the continuous time Bellman's "nonlinear" second order partial differential equation with gradient constraint.

4.1   A Constrained Problem

For numerical implementation, we need to restrict the domain of definition of the state variables to a finite region $Q_M \equiv [0, M_W] \times [0, M_y] \times [0, M_z]$. For this purpose, we introduce the reflection processes L_t, R_t and D_t at the boundaries of the region. We hence consider the following modified state variables:

$$
\begin{aligned}
d\tilde{y}_t &= -\lambda \tilde{y}_t dt + \lambda dC_t - dL_t, \\
d\tilde{z}_t &= -\beta \tilde{z}_t dt + \beta dC_t - dR_t, \quad \text{and} \\
d\tilde{W}_t &= \tilde{W}_t(r + A_t(\mu - r))dt - dC_t + \tilde{W}_t A_t \sigma dB_t - dD_t,
\end{aligned}
$$

where L_t, R_t and D_t are nondecreasing and increase only when the state processes (W, z, y) hit the cutoff boundaries $W = M_W$, $z = M_z$, and $y = M_y$, respectively.

We recall that the stock price process follows a geometric Brownian motion and that the riskless interest rate is constant. We will, hence, apply the Markov chain approach

to solve numerically, on the restricted domain, the following infinite horizon program:

$$J^M(W, y, z) \equiv \sup_{A,C \in \mathcal{AC}} E \int_0^\infty e^{-\delta t} u(\tilde{y}_t, \tilde{z}_t) dt \tag{26}$$

subject to the dynamics of $\tilde{z}$ and $\tilde{y}$ and the dynamic budget constraint, where $(W, y, z) \in Q_M$ is the initial state. The Bellman's equation for this dynamic program takes the form:

$$\max\left\{-\delta J^M + \max_A\{\mathcal{L}^A J^M\} + u(y, z), \lambda J_y^M + \beta J_z^M - J_W^M\right\} = 0, \tag{27}$$

on $[0, M_W] \times [0, M_y] \times [0, M_z]$, together with the appropriate boundary conditions, where the operator $\mathcal{L}^A$ is given by $\mathcal{L}^A = \frac{1}{2}W^2 A^2 \sigma^2 \frac{\partial^2}{\partial W^2} + W[r + A(\mu - r)]\frac{\partial}{\partial W} - \lambda y \frac{\partial}{\partial y} - \beta z \frac{\partial}{\partial z}$. In addition, we specify the boundary constraints

$$\frac{\partial J^M}{\partial W} = 0, \qquad \frac{\partial J^M}{\partial y} = 0, \qquad \text{and} \qquad \frac{\partial J^M}{\partial z} = 0,$$

respectively at the cutoff boundaries $W = M_W$, $y = M_y$, and $z = M_z$. This type of boundary conditions was suggested in Kushner and Dupuis (1992, §5.7) and employed in the numerical analysis of the one–dimensional financial model studied by Fitzpatrick and Fleming (1992). These boundary constraints reflect the asymptotic behavior of the value function on the original unbounded domain. It is worthwhile to remark that in the theoretical analysis of convergence of the numerical scheme, the value functions J^M converge pointwise to J as the sequence of restricted domains Q_M increases to Q, *regardless of* the specification of the behavior of J^M at the boundaries. The boundary conditions we chose, however, affect the quality of the actual numerical solution.

4.2 The Markov Chain Scheme

We introduce a sequence of grids

$$G_h = \{(k,)) : W = k \times h, y = i \times h, z = j \times h; 0 \le k \le N_W, 0 \le i \le N_y, 0 \le j \le N_z\},$$

where h is the step size, and where $N_y = M_y/h$, $N_z = M_z/h$, and $N_W = M_W/h$ are integers. Each grid point $(k, i, j) \in G_h$ corresponds to a state (W, y, z) with $W = k \times h$,

$y = i \times h$, and $z = j \times h$. For simplicity, we take the same step size in all three directions. For a fixed grid, we introduce the space of discrete strategies

$$\mathcal{AC}_N = \{(A, C); A = l \times \Delta, \Delta C = 0 \text{ or } h; 0 \le l \le N_A\}$$

where Δ is the step size of control, $N_A = \bar{A}/\Delta$ is total number of control steps with $\bar{A}$ an artificial bound to be relaxed in the limit as $\Delta \downarrow 0$ and $N_A \times \Delta \uparrow \infty$.

We approximate the continuous time process $(\tilde{W}, \tilde{z}, \tilde{y})$ by a sequence of discrete parameter Markov chains $\{(W_n^h, y_n^h, z_n^h); n = 1, 2, \cdots\}$ with h denoting the granularity of the grid and n denoting the steps of the Markov chain. For every h, the chain has the property that, at each step n, there is a choice between investment and consumption. At any time, a chain can either make an instantaneous jump, $(\Delta C = h)$, or follow a "random walk", $(\Delta C = 0)$, to the neighboring states on the grid. At the cutoff boundaries the chain is reflected to the interior in a manner consistent with its dynamics in the interior of the domain. Fix a chain and its corresponding grid size h. The transition probabilities for this Markov chain are specified as follows:

1. The case of no consumption—$(\Delta C = 0)$:
The chain can possibly move from the current state (k, i, j) only to one of the four neighboring states: $(k + 1, i, j)$, $(k - 1, i, j)$, $(k, i - 1, j)$, and $(k, i, j - 1)$. For any investment policy $A = l \times \Delta$, the transition probabilities in this case are defined as

$$P_h^A[(k, i, j), (k + 1, i, j)] = \frac{k^2 h^2 A^2 \sigma^2 / 2 + k h^2 (r + \mu A)}{Q^h(k, i, j)},$$

$$P_h^A[(k, i, j), (k - 1, i, j)] = \frac{k^2 h^2 A^2 \sigma^2 / 2 + k h^2 (rA)}{Q^h(k, i, j)},$$

$$P_h^A[(k, i, j), (k, i - 1, j)] = \frac{\lambda i h^2}{Q^h(k, i, j)},$$

$$P_h^A[(k, i, j), (k, i, j - 1)] = \frac{\beta j h^2}{Q^h(k, i, j)}, \quad \text{and}$$

$$P_h^A[(k, i, j), (k, i, j)] = 1 - P_h^A[(k, i, j), (k + 1, i, j)] - P_h^A[(k, i, j), (k - 1, i, j)]$$
$$- P_h^A[(k, i, j), (k, i - 1, j)] - P_h^A[(k, i, j), (k, i, j - 1)]$$

where the normalization factor $Q^h(k,i,j)$ is given by

$$Q^h(k,i,j) = \max_{0 \le l \le N_A} \{k^2 h^2 (l \times \Delta)^2 \sigma^2 + kh^2[r + l \times \Delta(\mu + r)] + (\lambda i + \beta j)h^2\}.$$

The recipe for these transition probabilities is a slightly modified version of those suggested by Kushner (1977). Note that the normalization factor is independent of the investment policy A. This specification implies that, at each step, the Markov–chain has a strictly positive probability of remaining at the same node.

Furthermore, we define the incremental difference $\Delta y_n^h \equiv y_{n+1}^h - y_n^h$, $\Delta z_n^h \equiv z_{n+1}^h - z_n^h$, and $\Delta W_n^h \equiv W_{n+1}^h - W_n^h$. The quantity $\Delta t^h(k,i,j)$ is equal to $h^2/Q^h(k,i,j)$ and interpreted, following Kusnher and Dupuis (1992), as the "interpolation time" for the Markov chain. Note that the interpolation time $\Delta t^h(k,i,j)$ is a random variable that varies from state to state. At step n of the chain, with the previously defined time scale Δt_n^h, one can verify that

$$\begin{aligned}
E_n^h[\Delta y_n^h] &= -\lambda y \Delta t_n^h - \Delta L_n^h, \\
E_n^h[\Delta z_n^h] &= -\beta z \Delta t_n^h - \Delta R_n^h, \\
E_n^h[\Delta W_n^h] &= W[r + A(\mu - r)]\Delta t_n^h - \Delta D_n^h, \text{ and} \\
E_n^h[\Delta W_n^h - E_n^h[\Delta W_n^h]]^2 &= W^2 A^2 \sigma^2 \Delta t_n^h + O(\Delta t_n^h),
\end{aligned} \quad (28)$$

where E_n^h denotes expectation conditional on the nth-time state (W_n^h, y_n^h, z_n^h), and where the reflecting processes ΔL_n^h, ΔR_n^h, and ΔD_n^h equal to the positive value h only when y_n^h, z_n^h, and W_n^h reach their respective boundaries. This implies that the first and second moments of the Markov chain approximate those of the continuous process $(\bar{W}_t, \tilde{y}_s, \tilde{z}_s)$. We call this property "local" consistency of the Markov chain.

2. The case of consumption— $(\Delta C = h)$:
The chain jumps along the direction $(-1, \lambda, \beta)$ from the current state (k,i,j) to the state $(k-1, i+\lambda, j+\beta)$. However, the later state, $(k-1, i+\lambda, j+\beta)$, is usually not on the grid except in the trivial case $\lambda = 1 = \beta$. For ease of programming, we take the intersection of the direction vector $(-1, \lambda, \beta)$ with the corresponding surface boundary and randomize between three grid points adjacent to the intersection point. We randomize in such a way that the expected random increment will be along the direction $(-1, \lambda, \beta)$ and of length equal to the distance from the starting state to the point of intersection.

Without loss of generality, we assume hereafter that $\lambda < 1 < \beta$. In this case, the intersection occurs inside a triangle spanned by the three grid points $(k, i, j+1)$, $(k-1, i, j+1)$, and $(k-1, i+1, j+1)$. The transition probabilities can be defined as:

$$P_h^C[(k, i, j), (k, i, j+1)] = \frac{\beta - 1}{\beta},$$

$$P_h^C[(k, i, j), (k-1, i, j+1)] = \frac{1 - \lambda}{\beta}, \text{ and}$$

$$P_h^C[(k, i, j), (k-1, i+1, j+1)] = \frac{\lambda}{\beta},$$

In this case, also, we can verify the property of "local" consistency:

$$E_n^h[\Delta y_n^h] = \lambda \Delta C, \qquad E_n^h[\Delta z_n^h] = \beta \Delta C, \qquad \text{and} \qquad E_n^h[\Delta W_n^h] = -\Delta C \qquad (29)$$

where $\Delta C = \frac{h}{\beta}$ is the increment in consumption. The quantities in the right–hand side of the equalities in (29) are the respective changes in the continuous time process corresponding to ΔC units of consumption.

4.3   The Markov–Chain Decision Problem

A policy (A, C) is admissible if it preserves the Markov property in that

$$\text{Prob}\left(\begin{array}{l} (W_{n+1}^h, y_{n+1}^h, z_{n+1}^h) = (W', y', z') \\ \text{conditional on } \left\{ \begin{array}{l} (W_n^h, y_n^h, z_n^h) = (W, y, z) \\ (W_k^h, y_k^h, z_k^h), \ k \leq n. \end{array} \right. \end{array} \right) = \left\{ \begin{array}{ll} P_h^A[(k, i, j), (k', i', j')], & \text{if } \Delta C = 0, \\ P_h^C[(k, i, j), (k', i', j')], & \text{if } \Delta C = h, \end{array} \right.$$

where $W = kh$ and $W' = k'h$, $y = ih$ and $y' = i'h$, $z = jh$ and $z' = j'h$. The control problem for the discrete parameter Markov chain is then to solve the program:

$$J^h(k, i, j) \equiv \max_{A, C} E_{k,i,j}^h \sum_{n=0}^{\infty} e^{-\lambda t_n^h} u(y_n^h, z_n^h) \Delta t_n^h, \qquad (30)$$

where $t_n^h = \sum_{0 \leq l \leq n} \Delta t_l^h$. Note, (30) is analogous to (26) in the sense that the sum in the former approximates the expected integral in the latter.

The discrete dynamic programming equation now takes the iterative form

$$J^h(k,i,j) = \max\Big\{ \textstyle\sum_{k',i',j'} P_h^C[(k,i,j),(k',i',j')]J^h(k',i',j'),$$
$$\max_{0\le l\le N_A}\{e^{-\delta\Delta t^h(k,i,j)}\textstyle\sum_{k',i',j'} P_h^A[(k,i,j),(k',i',j')]J^h(k',i',j')\} \quad (31)$$
$$+U(i,j)\Delta t^h(k,i,j)\Big\}$$

for $(k,i,j)\in G_n$, with iterative reflection[6] at the artificial boundaries, where we recall that $P_h^A[*,*]$ and $P_h^C[*,*]$ are the transition probabilities of the chain. Now, let us denote

$$D_i^- J(k,i,j) = \frac{J(k,i,j)-J(k,i-1,j)}{h}, \qquad D_j^- J(k,i,j) = \frac{J(k,i,j)-J(k,i,j-1)}{h},$$

$$D_i^+ J(k,i,j) = \frac{J(k,i+1,j)-J(k,i,j)}{h}, \qquad D_j^+ J(k,i,j) = \frac{J(k,i,j+1)-J(k,i,j)}{h},$$

$$D_k^+ J(k,i,j) = \frac{J(k+1,i,j)-J(k,i,j)}{h}, \qquad D_k^- J(k,i,j) = \frac{J(k,i,j)-J(k-1,i,j)}{h}, \text{ and}$$

$$D_k^2 J(k,i,j) = \frac{J(k+1,i,j)-2J(k,i,j)+J(k-1,i,j)}{h^2}.$$

Using this notation, we can express the discrete Bellman's equation (31) in the form:

$$-\Big[\frac{1-e^{-\delta\Delta t^h(k,i,j)}}{\Delta t^h(k,i,j)}\Big]J^h + \max_A\{\mathcal{L}_h^A J^h(k,i,j)\} + U(i,j) \le 0, \qquad (32)$$

$$\lambda D_i^+ J^h(k-1,i,j+1) + \beta D_j^+ J^h(k,i,j) - D_k^- J^h(k,i,j+1) \le 0, \text{ where} \qquad (33)$$

$$\mathcal{L}_h^A = \frac{1}{2}W^2 A^2\sigma^2 D_k^2 + W(r+A\mu)D_k^+ - WArD_k^- - \lambda y D_i^- - \beta z D_j^-.$$

Furthermore, one of these two inequalities must hold as an equality at each $(k,i,j)\in G_h$. In particular, if it is optimal not to consume at (k,i,j) then (32) holds as an equality. Otherwise, (33) holds as an equality at (k,i,j). Clearly, $[1-e^{-\delta\Delta t^h(k,i,j)}]/\Delta t^h(k,i,j)$ approximates the discount factor δ as $h\downarrow 0$. This leads to a pair of discrete differential inequalities which gives a finite–difference approximation of Bellman's equation (31) for the constrained maximization problem on the domain $[0,M_W]\times[0,M_y]\times[0,M_z]$. In section 5, we outline a proof that the above described sequence of finite–difference approximations, together with the boundary specifications, converges to the solution of the continuous time Bellman's equation.

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[6]This specification will be detailed in the end of this section.

4.4 Boundary Specification of The Value Function

For numerical implementation, we attach a "fictitious" surface to each cutoff boundary surrounding the cubic grid G_h in which the Markov chain is defined. We denote by G_h^+ the grid including this attached exterior surfaces. The three surfaces ($W = 0, z = 0$, and $y = 0$) are natural boundaries whereas the three surfaces ($W = M_W + h, z = M_z + h$, and $y = M_y + h$) are artificial fictitious boundaries. The value function on the artificial boundaries will be specified to capture the optimal policy in a manner locally consistent with that of the continuous processes at the boundary surfaces. These are the conditions we turn to next.

- At the boundary $W = 0$, the boundary condition is given by the natural constraint $J^h(0, i, j) = \Psi(ih, jh)$.

- At the boundary surface $y = 0$, it is clear that the optimal policy calls for consumption because of infinite marginal utility for y on that boundary. Hence, the value function at this boundary is given by:

$$J^h(k, 0, j) = \frac{\beta - 1}{\beta} J^h(k, 0, j + 1) + \frac{1 - \lambda}{\beta} J^h(k - 1, 0, j + 1) + \frac{\lambda}{\beta} J^h(k - 1, 1, j + 1).$$

- Similarly, at the boundary $z = 0$, because of infinite marginal utility for z, immediate consumption is optimal. Hence, the value function satisfies:

$$J^h(k, i, 0) = \frac{\beta - 1}{\beta} J^h(k, i, 1) + \frac{1 - \lambda}{\beta} J^h(k - 1, i, 1) + \frac{\lambda}{\beta} J^h(k - 1, i + 1, 1).$$

Next consider the artificial cutoff boundaries.

- At the top-level boundary $W = M_W + h$, we take it that the wealth level is so high that the agent will optimally choose to consume regardless of the levels of z or y. As a result, the iterative scheme of the value function at that boundary is given by the natural constraint $\lambda D_i^+ J^h + \beta D_j^+ J^h - D_k^+ J^h = 0$.

- At the y-side boundary $y = M_y + h$, we let

$$D_i^+ J^h = 0, \quad \text{if } \Delta C = 0 \text{ and}$$
$$\beta D_j^+ J^h - D_k^+ J^h = 0, \quad \text{if } \Delta C = h.$$

- At the z-side boundary $z = M_z + h$, we let

$$D_j^+ J^h = 0, \quad \text{if } \Delta C = 0 \text{ and}$$
$$\lambda D_i^+ J^h - D_k^+ J^h = 0, \quad \text{if } \Delta C = h.$$

5 Convergence of the Numerical Scheme

In this section, we analyze the convergence of the value function and the optimal controls in the approximating Markov chain problems to those in the original continuous time problem. We first analyze the properties of the finite–difference approximation scheme described in section 4. In particular, we show that the discrete Markov chain approximation approach produces a monotone, stable, and consistent finite–difference approximation to the Bellman's equation that characterizes the value function.

Armed with these results, we proceed to prove that, as the mesh size goes to zero and the size of the cubic domain Q_M increases, the sequence of value functions in the discrete Markov chain problems converges to the value function of the original problem uniformly on any compact subset of the domain Q. Furthermore, convergence obtains regardless of the boundary conditions specified on the artificial boundaries of the restricted domains Q_M. The proof is based on the fact, we already established, that the value function J is the unique viscosity solution of Bellman's equation (27). In the classical case with a smooth value function (see, for instance, Strikwerda (1989)), convergence follows directly from consistency and stability of the finite–difference scheme. Here, the situation is different because we do not, in general, have a classical solution of Bellman's equation. We, hence, utilize the comparison principle for subsolutions and supersolutions of Bellman's nonlinear partial differential equation in order to guarantee convergence of the approximations to the value function of the continuous–time problem.

5.1 Properties of the Finite–Difference Scheme

Let $\mathcal{B}(G_h)$ denote the space of real–valued functions on the extended lattice G_h. To study the discrete Bellman's equation (31), we introduce the mappings

$$T_h^A : \ \mathcal{B}(G_h) \to \mathcal{B}(G_h) \qquad \text{and} \qquad S_h : \ \mathcal{B}(G_h) \to \mathcal{B}(G_h) \quad \text{where}$$

$$(T_h^A V)(k,i,j) \ \equiv \ e^{-\delta \Delta t^h(k,i,j)} \sum_{k',i',j'} P_h^A[(k,i,j),(k',i',j')]V(k',i',j'), \qquad \forall A$$

$$(S_h V)(k,i,j) \ \equiv \ \sum_{k',i',j'} P_h^C[(k,i,j),(k',i',j')]V(k',i',j'),$$

for any $V \in \mathcal{B}(G_h)$. In the case of no consumption, we set $\Delta_h = \min_{k,i,j} \Delta t^h(k,i,j)$. Obviously $\Delta_h > 0$. Define the operator norm $\|\cdot\|$ as $\|T\| = \sup_{v \in \mathcal{B}} \|Tv\|_\infty / \|v\|_\infty$, with $\|\cdot\|_\infty$ being the L^∞-norm of the bounded function space $\mathcal{B}$.

The discrete Bellman's equation can be rewritten as follows:

$$J^h(k,i,j) = \max\Big\{\max_A[T_h^A J^h(k,i,j)] + u(i,j)\Delta t^h(k,i,j), S_h J^h(k,i,j)\Big\}, \quad \forall(k,i,j) \in G_h. \tag{34}$$

It is clear that the solution of the discrete Bellman's equation is a fixed point of the operator equation (34). The properties of the operators T_h^A and S_h, which are useful for proving convergence, are recorded in the following proposition.

Proposition 2 T_h^A's *are strict contraction operators with* $\|T_h^A\| \le e^{-\delta \Delta_h} < 1$ *and* S_h *is a contraction operator with* $\|S_h\| \le 1$. *Thus, for every h, there exists a function* J^h *that solves (34).*

Proof. We estimate directly

$$|T_h^A V(k,i,j) - T_h^A V'(k,i,j)| \ \le \ e^{-\delta \Delta t^h(k,i,j)} \sum_{k',i',j'} P_h^A[(k,i,j),(k',i',j')]|V(k',i',j') - V'(k',i',j')|$$

$$\le \ e^{-\delta \Delta_h} \max_{(k,i,j) \in G_h} |V(k,i,j) - V'(k,i,j)| \quad \text{and}$$

$$|S_h V(k,i,j) - S_h V'(k,i,j)| \ \le \ \sum_{k',i',j'} P_h^C[(k,i,j),(k',i',j')]|V(k',i',j') - V'(k',i',j')|$$

$$\le \ \max_{(k,i,j) \in G_h} |V(k,i,j) - V'(k,i,j)|$$

which implies the property of contraction. Given that both T_h^A and S_h are both contractions, the second assertion follows. ∎

Next, we discuss the properties of the finite–difference scheme, (32) and (33), associated with the Markov chain approximation. On a grid of size h, we will denote a given finite–difference operator by $\mathcal{L}_h$ and interpret $\mathcal{L}_h v$ to be the result, at each grid point, of applying the scheme to the function v. We will that $v \leq v'$ if $v(W, y, z) \leq v'(W, y, z)$ at all $(W, y, z) \in Q$. Recall that a finite difference scheme is said to be monotone if $\mathcal{L}_h v \leq \mathcal{L}_h v'$ when $v \leq v'$. The scheme $\mathcal{L}_h v = f$ is said to be consistent with the partial differential equation $\mathcal{L}v = f$ if, for any smooth function ϕ, we have $\mathcal{L}_h \phi - \mathcal{L}\phi \to 0$, pointwise at each grid point, as $h \downarrow 0$. Finally, a finite–difference scheme $\mathcal{L}_h v^h = f$ is said to be stable if there exists $h_0 > 0$ such that $\|v^h\|_\mathcal{D} \leq K$, for all $0 < h < h_0$, for any compact domain $\mathcal{D}$ and some positive constant K independent of h, where $\|\cdot\|_\mathcal{D}$ denotes the $L^\infty(\mathcal{D})$–norm. The following result establishes the properties of the finite–difference scheme we constructed in section 4.2.

Theorem 3 *The discrete Markov chain approximation yields a monotone, stable, and consistent finite–difference scheme for Bellman's equation.*

Proof. First, monotonicity of the scheme is implied by monotonicity of the discrete operators $T_h^A V \leq T_h^A V'$ and $S_h V \leq S_h V'$ for any $V, V' \in \mathcal{B}(G_h)$ such that $V \leq V'$. Second, we can verify directly that, for any smooth function ϕ, and with the obvious notation:

$$
\begin{aligned}
T_h^A \phi^h(k,i,j) - \phi^h(k,i,j) &= -[1 - e^{-\delta \Delta t^h(k,i,j)}]\phi^h(k,i,j) + \mathcal{L}_h^A \phi^h(k,i,j)\Delta t^h(k,i,j), \\
S_h \phi^h(k,i,j) - \phi^h(k,i,j) &= [\lambda D_i^+ \phi^h(k-1,i,j+1) + \beta D_j^+ \phi^h(k,i,j) - D_k^- \phi^h(k,i,j+1)]h,
\end{aligned}
\tag{35}
$$

where $\phi^h(k,i,j) \equiv \phi(kh, ih, jh)$. The difference terms can be estimated by

$$
D_k^\pm \phi = \phi_W \pm \frac{1}{2}\phi_{WW} \times h + O(h^2), \qquad D_k^2 \phi = \phi_{WW} + O(h^2),
$$

$$
D_i^- \phi = \phi_y + \frac{1}{2}\phi_{yy} \times h + O(h^2), \text{ and} \qquad D_j^- \phi = \phi_z + \frac{1}{2}\phi_{zz} \times h + O(h^2).
$$

Therefore, $\mathcal{L}_h^A \phi - \mathcal{L}^A \phi = O(h)$, which implies the consistency of our finite–difference approximation. Finally, it follows from Proposition 2 that, for any investment policy A, the set $\{(T_h^A, S_h); \forall h\}$ is uniformly bounded in operator norm, which implies the stability of the finite–difference scheme. ∎

Note that consistency as used in theorem 3 is actually equivalent to the "local" consistency property (28) and (29) of the corresponding Markov chain.

5.2   Convergence of The Value Function

In this section, we report the result that the sequence of value functions in the approximating decision problems converges to the value function of the original continuous time program. The proof of this result requires special attention to the gradient constraints in the sequence of Bellman's equations of the approximating control problems.

Theorem 4 (Convergence) *The sequence of solutions $J^h(k, i, j)$ of the discrete Markov chain problems converges to the value function $J(W, y, z)$ of the continuous–time problem uniformly on any compact subset of $Q = (0, \infty)^3$ as $h \downarrow 0$ and $N_W \uparrow \infty, N_z \uparrow \infty$, and $N_y \uparrow \infty$, in such a way that $hk \to W, hi \to y$, and $hj \to z$.*

Proof.   For each $(W, y, z) \in Q$, define

$$J^*(W, y, z) \equiv \limsup_{\substack{(W', y', z') \to (W, y, z) \\ h \downarrow 0, (W', y', z') \in G^h}} J^h(W', y', z'), \quad \text{and} \tag{36}$$

$$J_*(W, y, z) \equiv \liminf_{\substack{(W', y', z') \to (W, y, z) \\ h \downarrow 0, (W', y', z') \in G^h}} J^h(W', y', z'). \tag{37}$$

J^* is finite at every point (W, y, z) since the finite–difference scheme is stable. Clearly, J^* is upper semicontinuous and J_* is lower semicontinuous, and

$$J_*(W, y, z) \leq J^*(W, y, z), \quad \text{on } Q. \tag{38}$$

Note that $J^h(0, y, z) = \Psi(y, z)$, $\forall (y, z) \in G^h$. Therefore, $J^*(0, y, z) = \Psi(y, z) = J_*(0, y, z)$ for $(y, z) \in (0, \infty) \times (0, \infty)$. We will verify here that J^* is a viscosity subsolution and J_*

is a viscosity supersolution of Bellman's equation

$$\max\left\{-\delta J + \max_A \{\mathcal{L}^A J\} + u(y,z), \lambda J_y + \beta J_z - J_W\right\} = 0, \quad \text{on } Q \quad \text{with } J(0,y,z) = \Psi(y,z).$$
(39)

It then follows from the comparison result in theorem 2 that $J^* \leq J_*$ on Q. This result, together with (38), implies that $J^* = J_*$, and hence, by uniqueness, both are equal to J.

Let $(W_0, y_0, z_0) \in Q$ be a strict local maximum of $J^* - \phi$ for some $\phi \in C^{2,1,1}(Q)$. Since we take the limit $h \downarrow 0$ and $(N_W, N_y, N_z) \uparrow \infty$ in such a way that $h \times (N_W, N_y, N_z) \uparrow \infty$, there exists some $h_0 > 0$ such that for $h < h_0$, (W_0, y_0, z_0) is an interior point of the domain spanned by the grid G_h. By the property of "limsup", there exists a sequence indexed by h such that

$$(W_h, y_h, z_h) \to (W_0, y_0, z_0), \qquad J^h(W_h, y_h, z_h) \to J^*(W_0, y_0, z_0) \qquad \text{as } h \downarrow 0.$$

For $h < h_0$, $J^h - \phi$ has a local maximum at (W_h, y_h, z_h), and hence, for some neighborhood $\mathcal{N}_0$ of (W_0, z_0, y_0),

$$J^h(W_h, y_h, z_h) - \phi(W_h, y_h, z_h) \geq J^h(W, y, z) - \phi(W, y, z), \quad \text{if } (W, y, z) \in \mathcal{N}_0. \quad (40)$$

Since J^h is a solution of the discrete operator equation

$$J^h(W_h, y_h, z_h) = \max\left\{\max_A [T_h^A J^h(W_h, y_h, z_h)] + u(y_h, z_h)\Delta t^h, S_h J^h(W_h, y_h, z_h)\right\},$$

using (40) and the monotonicity of T_h^A and S_h, we can replace J^h by ϕ such that

$$\max\left\{\max_A [T_h^A \phi - \phi](W_h, y_h, z_h) + u(y_h, z_h)\Delta t^h, [S_h\phi - \phi](W_h, y_h, z_h)\right\} \geq 0. \quad (41)$$

On the other hand, we recall from (35) that

$$T_h^A \phi - \phi = -[1 - e^{-\delta\Delta t^h}]\phi + \mathcal{L}_h^A \phi \Delta t^h, \quad \text{and} \quad (42)$$

$$S_h \phi - \phi = [\lambda D_i^+ \phi + \beta D_j^+ \phi - D_k^- \phi]h. \quad (43)$$

Therefore, dividing (42) by Δt^h and (43) by h, the inequality (41) at (W_h, y_h, z_h) becomes

$$\max\left\{-\left(\frac{1 - e^{-\delta\Delta t^h}}{\Delta t^h}\right)\phi + \max_A [\mathcal{L}_h^A \phi] + u(y_h, z_h), \lambda D_i^+ \phi + \beta D_j^+ \phi - D_k^- \phi\right\} \geq 0,$$

for $h < h_0$. Finally, by the consistency of our finite–difference scheme, as we let $h \downarrow 0$, we obtain

$$\max\left\{-\delta\phi + \max_A[\mathcal{L}^A\phi] + u(y_0, z_0), \lambda\phi_y + \beta\phi_z - \phi_W\right\} \geq 0, \quad \text{at } (W_0, y_0, z_0).$$

Thus, J^* is a viscosity subsolution by Definition 1.

Using similar reasoning, we can show that J_* is a viscosity supersolution. By the comparison principle, theorem 2, it follows that $J^*(W, y, z) \leq J_*(W, y, z)$ on Q which, combined with (38), proves that, for all $(W, y, z) \in Q$,

$$\lim_{\substack{(W',y',z')\to(W,y,z)\\ h\downarrow 0,\, (W',y',z')\in G^h}} J^h(W', y', z') \text{ exists} = J(W, y, z) = J^*(W, y, z) = J_*(W, y, z). \quad \blacksquare$$

In the argument of convergence, we do not require knowledge of the particular specifications on the cutoff boundaries for the Markov chains. The same limit for J^h will be obtained regardless of the conditions chosen at the cutoff boundaries. This results from the fact that, in the *theoretical* sequence of approximations, as the mesh size decreases, the size of the corresponding bounded domain increases. However, in the actual numerical implementation, the size of the restricted domain is kept fixed and hence the choice of the boundary conditions affects the quality of the numerical solution. The boundary specifications discussed in the previous sections were chosen to obtain a good approximation to the solution of the continuous–time problem.

5.3   Convergence of Approximating Policies

Existence of the value function and the feasibility of computing its values with high accuracy do not guarantee existence of optimal policies that achieve the value function. Contrast this with classical feedback control case. In that case, optimal controls can be expressed as explicit functions of the value function and its derivatives. As a consequence, existence of a smooth value function immediately implies existence of optimal controls. In our case, with a free–boundary problem, there is no direct relationship between the value function and the optimal controls, which, if exist, are not feedback. Furthermore, the value function is not known to be smooth.

¨ In this section, we prove convergence of the optimal investment policies in the approximating Markov chains to the optimal investment policy of the diffusion control problem if the latter indeed exists. We will parameterize the sequence of approximating problems by h and denote the optimal consumption and investment policies in problem h by C^{*h} and A^{*h}, respectively. We define an interpolation of the value function for the Markov chain problem using the following linear affine interpolation:

$$
\begin{aligned}
\bar{J}^h(W, y, z) \equiv\ & J^h(k, i, j) + D_k^+ J^h(k, i+1, j+1) \times (W - kh) \\
& + D_i^- J^h(k, i, j+1) \times (y - ih) + D_j^- J^h(k, i, j) \times (z - jh)
\end{aligned}
$$

for $kh \le W < (k+1)h$, $ih \le y < (i+1)h$, and $jh \le W < (j+1)h$; with appropriate extensions at the cutoff boundaries. Clearly, $\bar{J}^h(W, y, z)$ is continuous on the cubic region $[0, M_W] \times [0, M_y] \times [0, M_z]$. One can easily show the following

Lemma 1 *As a function of (W, y, z), $\bar{J}^h(W, y, z)$ is nondecreasing in (W, y, z) and concave in W on $Q_M = [0, M]^3$.*

Proof. The proof that $\bar{J}^h$ is nondecreasing in its arguments follows the same reasoning as the proof of property 1 of the value function in Fitzpatrick and Fleming (1991, page 827). Furthermore, the proof of concavity of $\bar{J}^h$ in W follows the proof of property 2 (page 829) in the same reference. ∎

Next, we utilize the special monotonicity and concavity features of the problem to prove convergence of the first and second order derivatives $\partial \bar{J}_W^h(k, i, j) \to J_W(W, y, z)$ and $\partial \bar{J}_{WW}^h(k, i, j) \to J_{WW}(W, y, z)$ as $h \downarrow 0$ and $(hk, hi, hj) \to (W, y, z)$. This information is used to show that the optimal investment A^h for the discrete Markov chain control problem converges to the optimal investment policy A^* for the continuous–time model when the latter exists.

Theorem 5 *Suppose that there exists an optimal control policy (A^*, C^*) for the original continuos time problem. Then, the sequence of policies $A^{*h}(k, i, j)$ converges at all points (W, y, z) where J_W and J_{WW} exist, which is almost everywhere, to the optimal investment*

policy $A^*(W, y, z)$ as $h \downarrow 0$, and $N_W \uparrow \infty$, $N_z \uparrow \infty$, and $N_y \uparrow \infty$, in such a way that $hk \to W$, $hi \to y$, and $hj \to z$.

Proof. Using standard arguments, we can show that the value function J is increasing and concave in its arguments. It then follows; see Royden (1988, §5.5), that J has first and second derivatives almost everywhere. As a result, since J is a viscosity solution of Bellman's equation, J satisfies that equation in almost everywhere sense. Carrying out the maximization in (7), we conclude that $A^* = \frac{-J_W}{J_{WW}W} \frac{\mu - r}{\sigma^2}$ almost everywhere.

We rewrite the "finite–difference" form of the Bellman's equation for the interpolated value function $\bar{J}^h$

$$[\frac{1 - e^{-\delta \Delta t^h}}{\Delta t^h}]\bar{J}^h(k, i, j) + O(h) = \max_{0 \le l \le N_A} \{\mathcal{L}_h^{l\Delta} \bar{J}^h(k, i, j)\} + U(i, j), \qquad (44)$$

in the case of no consumption $\Delta C = 0$. From this, we can compute the optimal investment policy by the formula $A^{*h}(k, i, j) = -\frac{\mu \times D_k^+ \bar{J}^h - r \times D_k^- \bar{J}^h}{W\sigma^2 D_k^2 \bar{J}^h}$, for $(k, i, j) \in G_h^0$ – the interior grid. Note that only the first and second order derivatives of $\bar{J}$, with respect to W, are used in that formula. Substituting the previous formula back into the difference equation (44), and after a long calculation, we can express $D_k^2 \bar{J}^h$ as $H(D_k^\pm \bar{J}^h, D_i^- \bar{J}^h, D_j^- \bar{J}^h, \bar{J}^h)$ for some continuous function H. Now, using arguments similar to those in Fitzpatrick and Fleming (1991, theorem 4), and lemma 1 , we can prove that $D_k^\pm \bar{J}^h \to J_W$, $D_i^- \bar{J}^h \to J_y$, and $D_j^- \bar{J}^h \to J_z$. As a result, $D_k^2 \bar{J}^h \to J_{WW}$ and consequently we must have that $A^{*h} \to A^*$, at the points of differentiability with respect to W, as $h \downarrow 0$ and as the size of the domain Q_M increases to Q. $\blacksquare$

Finally, we provide the following result that shows that the optimal "regions" of consumption and abstinence in the approximating sequence converge to the corresponding regions in the original problem. Specifically, if at a given point $(\bar{W}, \bar{y}, \bar{z})$, the optimal solution of the original problem prescribes a jump in consumption, then, all the approximating optimal solutions, except for, at most, a finite number, also prescribe jumps at the grid points that converge to $(\bar{W}, \bar{y}, \bar{z})$. Similarly, if the optimal solution in the continuous time problem calls for no consumption at $(\bar{W}, \bar{y}, \bar{z})$, then almost all the approximating

optimal solutions will also call for no consumption at the grid-points that converge to $(\bar{W}, \bar{y}, \bar{z})$.

Theorem 6 *Suppose that there exists an optimal control policy* (A^*, C^*) *for the original continuos time problem. Consider the sequence of approximating problems indexed by* h *and let* $h \downarrow 0$, *and* $N_W \uparrow \infty, N_z \uparrow \infty,$ *and* $N_y \uparrow \infty.$ *Let* $\bar{x} \equiv (\bar{W}, \bar{y}, \bar{z}) \in Q$ *be such that the value function* J *is twice continuously differentiable in a neighborhood of* $\bar{x}.$ *Let* $\Delta C^*(\bar{x})$ *be the size of the optimal consumption jump at* $\bar{x}.$ *Let* $\{\bar{x}^h\} \equiv \{(k, i, j)\}$ *be a sequence of grid points that converges to* $\bar{x}$ *in such a way that* $J^h(\bar{x}^h) \to J(\bar{x}).$ *If* $\Delta C^*(\bar{x}) > 0$ *(* $\Delta C^*(\bar{x}) = 0$ *), then* $\Delta C^{*h}(\bar{x}^h) > 0$ *(* $\Delta C^{*h}(\bar{x}^h) = 0$ *), respectively, for all* h, *except for, at most, a finite number.*

Proof. Consider the case when $\Delta C^*(\bar{x}) = 0$. Since J is smooth in a neighborhood of $\bar{x}$, it follows from arguments similar to those used for proving the verification result—theorem 2— in HH&Z that

$$\beta J_z + \lambda J_y - J_W < 0 \quad \text{at } \bar{x}. \tag{45}$$

Now suppose that $\Delta C^{*h}(\bar{x}^h) = 0$ for only a finite number of points in the sequence. It then follows from Bellman's equation for the Markov chain decision problem that $\beta D_j^+ J^h + \lambda D_i^+ J^h - D_k^+ J^h = 0$ at $\bar{x}^h$, for all but a finite number of elements in the sequence. However, from the proof of theorem 5, we conclude that

$$\lim_{h \downarrow 0} (\beta D_j^+ J^h + \lambda D_i^+ J^h - D_k^+ J^h)(\bar{x}^h) = (\beta J_z + \lambda J_y - J_w)(\bar{x}) = 0$$

which contradicts (45). The proof of the case when $\Delta C^*(\bar{x}) > 0$ follows the same arguments and utilizes the result of theorem 5 that $\{A^{*h}\}$ converge to A^*. ∎

6   Concluding Remarks

The analysis of this paper achieves its objective of providing technical support for the numerical solution reported in HH&Z. In particular, we provide proofs of convergence of the sequences of both the value functions, the optimal investment policies, and the

. . optimal consumption regions in the approximating problems to the corresponding entities in the original problem. Such results justify the description in HH&Z of the numerical results as ϵ-optimal solutions. There are still some interesting theoretical open questions. We have not proved existence of an optimal solution in the class of admissible controls. Furthermore, we are also quite silent on the regularity properties of the free–boundary. We hope that further research in this area sheds some light on these issues.

We remark that there is another method of proving convergence of the sequence of value functions in the approximating Markov decision problems. That method is based on probabilistic techniques from the theory of weak convergence of probability measures. For a detailed exposition of the probabilistic approach, we refer the reader to Kushner and Dupuis (1992, §9 − §10) and the references therein. The idea there is, very roughly, as follows. One constructs continuous time interpolations of the optimally controlled Markov chains. Then one proves that the sequence of interpolated processes has a subsequence that converges weakly to an optimally controlled process of the original problem. Since the approximating sequence has the same objective functional as the original problem, one then concludes, using the theory of weak convergence, that the sequence of approximating value functions converges to the value function of the original problem.

Furthermore, by extending the space of admissible controls to the so–called "relaxed controls"; see Kushner and Dupuis (1992, §4.6), one can show that there exists an optimal relaxed control for the original problem. The optimal relaxed controls, however, need not be in the class of original admissible controls. In any case, the sequence of approximating optimal controls converges weakly to the optimal relaxed control of the original problem. We chose not to present the probabilistic approach here because the notion of weak convergence does not guarantee that the optimal relaxed control are adapted to the information structure in the economy. As a result, the optimal relaxed control may fail to be feasible and admissible control policies.

We remark, however, that using the analytical approach we can prove that the approximating optimal investment policies and the consumption regions converge pointwise to their corresponding counterparts in the original problem if the latter exists in the class of admissible controls. This is a stronger result than weak convergence to a relaxed con-

trol, which may not be in the class of the original admissible controls, that obtains using probabilistic techniques. Finally, we note that neither approach establishes existence of optimal policies in the original admissible class of controls described in section 2.2. Establishing such a result requires information about the regularity of the free–boundary. For more on this issue, we refer the reader to Soner and Shreve (1989) and Hindy and Huang (1993).

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